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ROLL WAVES IN A GAS–LIQUID MEDIUM

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The one-velocity motion of a gas–liquid medium with a variable mass fraction of the gas phase, which is equilibrium in terms of phase pressures, is considered. The existence conditions of nonlinear periodic wave packets similar in structure to roll waves in open inclined channels are found. The structure of travelling waves in the medium with continuous addition of energy to the gas phase is studied.

Introduction. The term "roll waves" is used in open-channel hydraulics to describe a quasi-periodic wave flow regime in inclined channels, where smooth parts of the flow are separated by breaking hydraulic jumps or bores [1]. A distinctive feature of such flows is the transition from subcritical to supercritical flow in a coordinate system moving with the wave. The mathematical theory of roll waves on an inclined plane was constructed in [2].

The development of a nonlinear quasi-periodic flow regime corresponding to generation of roll waves from an unstable uniform flow is typical of a wide class of motions in heterogeneous media. In the case of a gas–liquid flow medium in horizontal channels and tubes, the development of flow instability leading to generation of finite-amplitude roll waves is one of the main mechanisms of transition from the stratified flow regime to the slug flow [3, 4]. The criterion of nonlinear stability of roll waves in one- and two-layer flows was proposed in [5].

A model of a gas–liquid medium with variable mass fraction of the gas component is constructed in the present paper. The model proposed is the simplest variant of a heterogeneous medium, where a self-oscillatory process in the form of periodic discontinuous travelling waves (roll waves) is excited due to continuous input of energy. The mechanical system considered is an analog of a more complicated gas–liquid medium, where nonlinear oscillations can be excited due to chemical reactions with heat release and phase transitions. Thus, conditions are created for constructing a medium, where mechanical energy is released as a result of a self-oscillatory process with continuous addition of internal energy ("acoustic laser" problem).

Mathematical Model of a Gas–Liquid Medium. We consider a one-velocity model of motion of a gas–liquid mixture. The carrier phase is an ideal incompressible liquid of density $\rho_{\rm liq}$. Let u be the velocity of the liquid, α be the volume fraction of the liquid phase, m be the gas mass per unit volume of the mixture, and $\rho = m/(1-\alpha)$ be the true density of the gas ($\rho \ll \rho_{\rm liq}$). The gas is assumed to be isothermal: $p_{\rm g}(\rho) = c_T^2\rho$, where $c_T \equiv$ const is the isothermal velocity of sound. Under the assumption that the pressures in the liquid and gas components are equal [$p = p_{\rm g}(\rho)$], the equations of motion take the form

$$\alpha_t + (\alpha u)_x = 0, \qquad u_t + (u^2/2 + p/\rho_{\rm liq})_x = 0,$$
$$(m\alpha)_t + (m\alpha u)_x = \alpha(\varphi(\alpha) - \mu p). \tag{1}$$

The last equation determines the mass balance of the gas phase with allowance of sources of intensity $\varphi(\alpha)$ [$\varphi(\alpha) > 0$, $\varphi'(\alpha) > 0$, and $\varphi''(\alpha) > 0$] and sink ($\mu =$ const). The behavior of the function $\varphi(\alpha)$ indicates that the rate of gas inflow increases with decreasing volume fraction of the gas phase.

In dimensionless variables, we may assume that $\rho_{\rm liq} = 1$ and $c_T = 1$. Besides, by additional extension of independent variables, we can ensure $\mu = 1$. System (1) is hyperbolic. There are two families of "sound" characteristics $dx/dt = \lambda^\pm = u \pm c$ and $c = \sqrt{m\alpha/(1-\alpha)}$ and a contact characteristic $dx/dt = \lambda_0 = u$. The conservation laws (1) yield the following relations on discontinuities:

$$[\alpha(D-u)] = 0, \qquad [(D-u)^2/2 + p] = 0, \qquad [\alpha m(D-u)] = 0. \tag{2}$$

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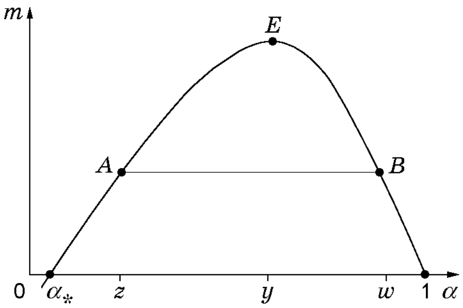

Fig. 1. Dependence $m = m(\alpha)$.

Here $D = dx/dt$ is the velocity of discontinuity propagation. We seek roll waves for system (1) in the class of periodic travelling waves.

Travelling Waves. We consider solutions of system (1) depending on the variable $\xi = x - Dt$ ($D \equiv \text{const}$). The travelling waves are described by the following equations:

$$(D-u)\alpha = q \equiv \text{const}, \qquad (D-u)^2/2 + m/(1-\alpha) = J \equiv \text{const},$$
$$(D-u)\,\frac{dm}{d\xi} = \frac{m}{1-\alpha} - \varphi(\alpha). \tag{3}$$

System (3) can be reduced to one equation

$$\frac{q}{\alpha}\,\frac{dm}{d\xi} = \frac{q\Delta}{\alpha}\,\frac{d\alpha}{d\xi} = F, \tag{4}$$

where $m = m(\alpha) = (J - q^2/(2\alpha^2))(1-\alpha)$, $\Delta = \Delta(\alpha) = (1-\alpha/2)q^2/\alpha^3 - J$, and $F = F(\alpha) = J - q^2/(2\alpha^2) - \varphi(\alpha)$.

For construction of periodic solutions with discontinuities, the subcritical flow behind the discontinuity front ($\Delta < 0$) should be transformed to a supercritical flow ahead of the next front ($\Delta > 0$) as the continuous solution of Eqs. (3). Therefore, similar to roll waves in open channels, there should exist a critical value $\alpha = y$ such that

$$\Delta(y) = 0. \tag{5}$$

The following condition should also be fulfilled for a continuous solution of system (3) satisfying (5) to exist:

$$F(y) = 0. \tag{6}$$

Note that there is the only critical point y on the interval (0, 1) since

$$\frac{d}{d\alpha}\,\Delta(\alpha) = -\frac{1}{2}\,\frac{q^2}{\alpha^3} - \Big(1 - \frac{\alpha}{2}\Big)\frac{3q^2}{\alpha^4} < 0.$$

Therefore, $\Delta(\alpha) > 0$ for $\alpha < y$ and $\Delta(\alpha) < 0$ for $\alpha > y$.

We consider travelling waves moving to the right ($D > u$). By virtue of (5) and (6), the parameters q and J are functions of one variable y:

$$q = \sqrt{y^3\varphi(y)/(1-y)}, \qquad J = (1-y/2)\varphi(y)/(1-y). \tag{7}$$

To construct a periodic travelling wave, it is sufficient to fix the value $\alpha = z$ ahead of the discontinuity, which transforms a supercritical flow $[0\Delta(z) > 0]$ to a subcritical flow with a volume fraction $\alpha = w$ [$z < w$ and $\Delta(w) < 0$]. It follows from the relations on the discontinuity (2) that $m(z) = m(w)$. For the existence of a continuous solution of Eq. (4) relating the states $\alpha = z$ and $\alpha = w$, the following conditions are necessary and sufficient:

$$F(\alpha) > 0 \quad \text{for} \quad z < \alpha < y, \qquad F(\alpha) < 0 \quad \text{for} \quad y < \alpha < w. \tag{8}$$

In this case, $d\alpha/d\xi > 0$, and the periodic solution corresponds to the transition ABEA shown in Fig. 1. The roll wave consists of the discontinuity AB and subsequent smooth section BEA, where the transition from the subcritical flow BE to the supercritical flow EA occurs. The structure of this wave is similar to the structure of roll waves in an open channel [2].

Relations (7) and (8) yield the necessary condition for the existence of roll waves:

$$F'(y) = \varphi(y)/(1-y) - \varphi'(y) < 0. \tag{9}$$

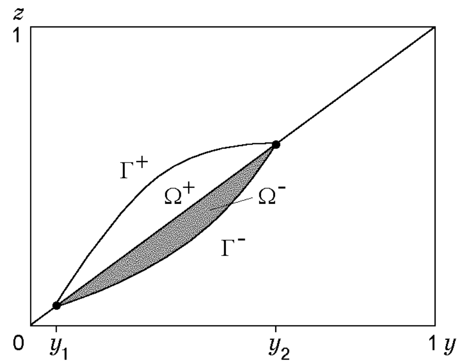

Fig. 2. Diagram of roll waves.

According to the theory of kinematic waves [1], condition (9) means that the propagation velocity of one characteristic $\lambda_e^{\pm}$ of the equilibrium model

$$\alpha_t + (\alpha u)_x = 0, \qquad u_t + (u^2/2 + \varphi(\alpha))_x = 0,$$

obtained from (1) for $\varphi(\alpha) \equiv \rho$ ($c_T = 1$, $\rho_{\rm liq} = 1$, and $\mu = 1$) is greater than the propagation velocity of the corresponding characteristic $\lambda^{\pm}$ for system (1). For waves moving to the right ($D > u$), this inequality takes the form

$$\lambda_e^+ = u + \sqrt{y\varphi'(y)} > u + \sqrt{\varphi(y)y/(1-y)} = \lambda^+. \tag{10}$$

Equations (9) and (10) are equivalent. Note, condition (9) is sufficient for the existence of infinitesimal-amplitude roll waves. If relations (5), (6), and (9) are satisfied, then we have $F(\alpha) < 0$ for $y < \alpha < 1$, since $F''(\alpha) = -3q^2/\alpha^4 - \varphi''(\alpha) < 0$. Therefore, condition (8) may be violated only on the interval (α_*, y), where $m(\alpha_*) = 0$ and $\alpha_* = \sqrt{q^2/(2J)} = \sqrt{y^3/(2-y)}$ (Fig. 1). In addition, we have $F(\alpha_*) = -\varphi(\alpha_*) < 0$, and there exists a unique value $\alpha = z_m$ on the interval (α_*, y), for which the function $F(\alpha)$ vanishes. As $z \to z_m$, the wave length tends to infinity, and a wave of the limiting amplitude is observed.

It follows from the above consideration that the roll-wave profile is determined by the parameters y and z ($0 < z_m < z < y < 1$) if condition (9) is satisfied. Owing to invariance of Eqs. (1) to the Galileo transform, the wave velocity D is also a free parameter.

Figure 2 shows the region of existence of roll waves (shaded region) in the plane (y, z) for the function $\varphi(\alpha) = 10\alpha^2 + 1$. Condition (9) is satisfied for $0 < y_1 < y < y_2 < 1$. The boundaries Γ^+ and Γ^- of the domain of existence of roll waves correspond to waves of the maximum amplitude. The curve Γ^- is defined by the equation $z = z_m(y)$, and the curve Γ^+ is prescribed by the equation $w = w_m(y)$; the maximum volume fraction of the liquid phase $w_m = w_m(y)$ in the roll wave is found from relations (2), i.e., $m(z_m) = m(w_m)$. For $y_1 < y < y_2$, the diagonal of the square corresponds to roll waves of infinitesimal amplitude.

For all values of the parameters y and z within the domain $\Omega^- = \{(y,z) : 0 < z_m < z < y,\ y_1 < y < y_2\}$ bounded by the curve Γ^- and diagonal of the square, there exists a roll wave with a minimum volume fraction of the liquid phase z and a maximum volume fraction w; the point (y, w) belongs to the domain $\Omega^+ = \{(y,z) : y < w < w_m < 1,\ y_1 < y < y_2\}$ located between the curve Γ^+ and the diagonal of the square (Fig. 2).

As for roll waves in an open inclined channel, the problem of stability of a packet of roll waves of a given amplitude is nontrivial and requires additional consideration. The criterion of nonlinear stability of roll waves, which was obtained in [5] by analysis of hyperbolicity of modulation equations, is applicable, in principle, to system (1), but this study is outside the scope of the present work.

Conclusions. A mechanical model of a gas–liquid medium is constructed. Under certain conditions, the energy added to gas cavities in this medium may be converted to the kinetic energy of motion of the mixture due to internal self-organization of the wave motion. An exact analogy can be found between the periodic solutions of system (1) constructed above and roll waves in inclined open channels. At the same time, there is an analogy between system (1) and the gas–liquid medium in which the energy release at the wave front is performed due to continuous addition of heat to the gas phase so that the heating rate increases with decreasing gas volume, and the cooling results from heat exchange with the carrier liquid phase. Thus, the simplest mathematical model is constructed for a gas–liquid medium with internal energy supply, where a nonlinear periodic wave process can be excited.

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